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A Weak Formulation of Roe's Scheme for Two-Dimensional, Unsteady, Compressible Flows and Steady, Supersonic Flows

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Abstract—A weak formulation of Roe's approximate Riemann solver developed recently for one-dimensional, unsteady, compressible flows is extended to two dimensions using operator splitting, and to steady, supersonic flows, and it is shown that the resulting schemes are analogous to other methods for such flows. The technique for steady flows draws on recent work in this area.

Keywords—Euler equations, Riemann solver, Weak formulation.

1. INTRODUCTION

In a recent paper, Toumi [1] presented a weak formulation of Roe's approximate Riemann solver based on a definition of a nonconservative product. Toumi first identifies the Lipschitz continuous path connecting two states that leads to the Roe-averaged state [2] for an ideal gas, and then constructs a generalised Roe-averaged matrix for the Euler equations in one-dimension with real gases by using the same path. In a recent paper [3], it is shown that employing the ideas presented in [1] to the simpler system of equations governing shallow water flows leads to a known approximate Riemann solver. In this paper, we extend and generalise Toumi's scheme to two dimensions for unsteady, compressible flows of an ideal gas, and to steady, supersonic flows, and show that the resulting schemes are also well-known. For the latter case, we draw on recent work on the interpretation of the weak formulation [4], but in addition show that the two approaches presented there are equivalent, and that because of this a simpler derivation is possible. In the future, we intend to generalise Toumi's method further to include two-dimensional flows of a real gas, as well as analysing the relationship between the weak formulation and parameter vector ideas, and the conditions for the resulting schemes to coincide.

2. COMPRESSIBLE FLOWS

The Euler equations of gas dynamics in two dimensions can be written as

$$\mathbf{u}_t + \mathbf{f}_x + \mathbf{g}_y = \mathbf{0}, \tag{2.1}$$

where

$$\mathbf{u} = (\rho, \rho u, \rho v, e)^\top, \tag{2.2}$$

$$\mathbf{f} = (\rho u, p + \rho u^2, \rho uv, u(e+p))^\top, \tag{2.3}$$

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$$\mathbf{g} = (\rho v, \rho vu, p + \rho v^2, v(e+p)), \quad \text{and} \tag{2.4}$$

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2). \tag{2.5}$$

The quantities $(\rho, u, v, e, p) = (\rho, u, v, e, p)(x, y, t)$ represent density, the two velocity components, total energy and pressure at a general position (x, y) in space and at time t, and where γ is the ratio of specific heat capacities of the fluid.

We consider the unsteady and steady cases separately since the solution technique is viewed differently in each case.

3. AN APPROXIMATE RIEMANN SOLVER (WEAK FORMULATION)

Unsteady case

In [1] it is proposed solving equations of the form (2.1) via locally linearised Riemann problems. The natural extension for two-dimensional flows is to use operator splitting, and solve a series of "one-dimensional" problems in the x-, and then in the y-direction, and this was considered first in [4] for the shallow water equations. Thus, for the system (2.1), a corresponding locally linearised Riemann problem in the x-direction along $y = y_0$, is

$$\mathbf{u}_t + A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}} \mathbf{u}_x = \mathbf{0}, \tag{3.1}$$

$$\mathbf{u}(x, y_0, 0) = \begin{cases} \mathbf{u}_L, & \text{if } x < 0, \\ \mathbf{u}_R, & \text{if } x > 0, \end{cases} \tag{3.2}$$

where $A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}}$ is a constant matrix which depends on the data $(\mathbf{u}_L, \mathbf{u}_R)$ and on the path $\mathbf{\Phi}(s; \mathbf{u}_L, \mathbf{u}_R)$, and satisfies

$$\int_0^1 A\left(\mathbf{\Phi}(s; \mathbf{u}_L, \mathbf{u}_R)\right) \frac{\partial \mathbf{\Phi}}{\partial s}(s; \mathbf{u}_L, \mathbf{u}_R)\, ds = A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}}(\mathbf{u}_R - \mathbf{u}_L), \tag{3.3}$$

$$A(\mathbf{u}, \mathbf{u})_{\mathbf{\Phi}} = A(\mathbf{u}), \quad \text{and} \tag{3.4}$$

$$A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}} \tag{3.5}$$

has real eigenvalues and a complete set of eigenvectors, where $A = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ is the Jacobian of $\mathbf{f}$. (N.B. This also applies to nonconservative systems of the form $\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0}$. However, when the system is conservative, as is the case here, (3.3) is equivalent to the condition $\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_R) = A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}}(\mathbf{u}_R - \mathbf{u}_L)$, which was originally proposed by Roe [2].)

As noted by Roe [2], the canonical path (a straight line) linking $\mathbf{u}_L$ and $\mathbf{u}_R$

$$\mathbf{\Phi}(s; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{u}_L + s(\mathbf{u}_R - \mathbf{u}_L), \qquad s \in [0, 1], \qquad \text{gives} \tag{3.6}$$

$$A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}} = \int_0^1 A\left(\mathbf{u}_L + s(\mathbf{u}_R - \mathbf{u}_L)\right) ds, \tag{3.7}$$

which will, in general, involve integrals which may not emerge in closed form, or the closed form may be expensive to compute. The alternative approach adopted by Roe is to introduce a parameter vector $\mathbf{w}$, and it is shown in [1] that the choice of the canonical path for $\mathbf{w}$ leads to Roe's original scheme for the Euler equations with ideal gases [2]. This choice is then employed in the case of real gases to lead to a new scheme [1].

The Riemann solver in [1] is constructed by letting $\mathbf{f}_0$ be a smooth function such that $\mathbf{f}_0(\mathbf{w}_L) = \mathbf{u}_L$, $\mathbf{f}_0(\mathbf{w}_R) = \mathbf{u}_R$ and $A_0(\mathbf{w}) = \partial \mathbf{f}_0 / \partial \mathbf{w}$ is a regular matrix for every state $\mathbf{w}$. The path chosen linking the two states $\mathbf{u}_L$ and $\mathbf{u}_R$ is then

$$\mathbf{\Phi}_0(s; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}_0\left(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)\right), \tag{3.8}$$

and this leads to the Roe matrix

$$A(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}\, B(\mathbf{u}_L, \mathbf{u}_R)^{-1}_{\Phi_0}, \qquad \text{where} \tag{3.9}$$

$$B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A_0\left(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)\right) ds, \qquad \text{and} \tag{3.10}$$

$$C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A\left(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\right) A_0\left(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)\right) ds, \tag{3.11}$$

which satisfies (3.3)–(3.5).

Our aim now is to show that the application of this Riemann solver to the two-dimensional equations of flow in Section 2 leads to the Riemann solver given in [2].

4. APPLICATION TO UNSTEADY, COMPRESSIBLE FLOWS

For equations (2.1)–(2.5), with parameter vector

$$\mathbf{w} = (w_1, w_2, w_3, w_4)^{\top} = (\sqrt{\rho}, \sqrt{\rho}u, \sqrt{\rho}v, \sqrt{\rho}H)^{\top}, \tag{4.1}$$

where $H = (e+p)/\rho$ is the total enthalpy, then

$$\mathbf{f}_0(\mathbf{w}) = \mathbf{u} = (\rho, \rho u, \rho v, e)^{\top} = \left(w_1^2, w_1 w_2, w_1 w_3, \frac{w_1 w_4}{\gamma} + \frac{(\gamma-1)}{2\gamma}\left(w_2^2 + w_3^2\right)\right)^{\top}, \tag{4.2}$$

so that

$$A_0 = \frac{\partial \mathbf{f}_0}{\partial \mathbf{w}} = \begin{pmatrix} 2w_1 & 0 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ w_3 & 0 & w_1 & 0 \\ \frac{w_4}{\gamma} & \frac{(\gamma-1)}{\gamma} w_2 & \frac{(\gamma-1)}{\gamma} w_3 & \frac{w_1}{\gamma} \end{pmatrix}. \tag{4.3}$$

From (3.10) and (4.3)

$$B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A_0\left(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)\right) ds = \begin{pmatrix} 2\bar{w}_1 & 0 & 0 & 0 \\ \bar{w}_2 & \bar{w}_1 & 0 & 0 \\ \bar{w}_3 & 0 & \bar{w}_1 & 0 \\ \frac{\bar{w}_4}{\gamma} & \frac{(\gamma-1)}{\gamma} \bar{w}_2 & \frac{(\gamma-1)}{\gamma} \bar{w}_3 & \frac{\bar{w}_1}{\gamma} \end{pmatrix}, \tag{4.4}$$

where the overbar denotes the arithmetic mean of left and right states, $\overline{\mathbf{w}} = \frac{1}{2}(\mathbf{w}_L + \mathbf{w}_R)$. To construct the matrix $C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}$ (having found $B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}$), and hence $A(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}$, it is necessary to write the Jacobian

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma-3)}{2}u^2 + \frac{(\gamma-1)}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ -uH + \frac{(\gamma-1)}{2}u^3 + \frac{(\gamma-1)}{2}uv^2 & H - (\gamma-1)u^2 & -(\gamma-1)uv & \gamma u \end{pmatrix}, \tag{4.5}$$

as a function of $\mathbf{w}$:

$$A(\mathbf{u}(\mathbf{w})) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma-3)}{2}\frac{w_2^2}{w_1^2} + \frac{(\gamma-1)}{2}\frac{w_3^2}{w_1^2} & (3-\gamma)\frac{w_2}{w_1} & -(\gamma-1)\frac{w_3}{w_1} & (\gamma-1) \\ -\frac{w_2 w_3}{w_1^2} & \frac{w_3}{w_1} & \frac{w_2}{w_1} & 0 \\ -\frac{w_2 w_4}{w_1^2} + \frac{(\gamma-1)}{2}\frac{w_2^3}{w_1^3} + \frac{(\gamma-1)}{2}\frac{w_2 w_3^2}{w_1^3} & \frac{w_4}{w_1} - (\gamma-1)\frac{w_2^2}{w_1^2} & -(\gamma-1)\frac{w_2 w_3}{w_1^2} & \frac{\gamma w_2}{w_1} \end{pmatrix}. \tag{4.6}$$

Combining (4.3) and (4.6) gives

$$A\left(\mathbf{u}(\mathbf{w})\right)A_0(\mathbf{w}) = \begin{pmatrix} w_2 & w_1 & 0 & 0 \\ \frac{(\gamma-1)}{\gamma}w_4 & \frac{(\gamma+1)}{\gamma}w_2 & -\frac{(\gamma-1)}{\gamma}w_3 & \frac{(\gamma-1)}{\gamma}w_1 \\ 0 & w_3 & w_2 & 0 \\ 0 & w_4 & 0 & w_2 \end{pmatrix}, \tag{4.7}$$

so that from (3.11)

$$\begin{aligned} C(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0} &= \int_0^1 A\left(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\right) A_0\left(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)\right)\,ds \\ &= \begin{pmatrix} \bar{w}_2 & \bar{w}_1 & 0 & 0 \\ \frac{(\gamma-1)}{\gamma}\bar{w}_4 & \frac{(\gamma+1)}{\gamma}\bar{w}_2 & -\frac{(\gamma-1)}{\gamma}\bar{w}_3 & \frac{(\gamma-1)}{\gamma}\bar{w}_1 \\ 0 & \bar{w}_3 & \bar{w}_2 & 0 \\ 0 & \bar{w}_4 & 0 & \bar{w}_2 \end{pmatrix}, \end{aligned} \tag{4.8}$$

where again $\bar{\mathbf{w}} = \frac{1}{2}(\mathbf{w}_L + \mathbf{w}_R)$ denotes the arithmetic mean.

Combining (4.4) and (4.8), we find that the matrix in (3.9) for the system of equations under consideration here is

$$\begin{aligned} A(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0} &= C(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0}\, B(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0}^{-1} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma-3)}{2}\frac{\bar{w}_2^2}{\bar{w}_1^2} + \frac{(\gamma-1)}{2}\frac{\bar{w}_3^2}{\bar{w}_1^2} & (3-\gamma)\frac{\bar{w}_2}{\bar{w}_1} & -(\gamma-1)\frac{\bar{w}_3}{\bar{w}_1} & \gamma-1 \\ -\frac{\bar{w}_2\bar{w}_3}{\bar{w}_1^2} & \frac{\bar{w}_3}{\bar{w}_1} & \frac{\bar{w}_2}{\bar{w}_1} & 0 \\ -\frac{\bar{w}_2\bar{w}_4}{\bar{w}_1^2} + \frac{(\gamma-1)}{2}\frac{\bar{w}_2^3}{\bar{w}_1^3} + \frac{(\gamma-1)}{2}\frac{\bar{w}_2\bar{w}_3^2}{\bar{w}_1^3} & \frac{\bar{w}_4}{\bar{w}_1} - (\gamma-1)\frac{\bar{w}_2^2}{\bar{w}_1^2} & -(\gamma-1)\frac{\bar{w}_2\bar{w}_3}{\bar{w}_1^2} & \frac{\gamma\bar{w}_2}{\bar{w}_1} \end{pmatrix}, \end{aligned} \tag{4.9}$$

and hence from (4.6), the approximate Jacobian matrix is

$$A(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0} = A(\mathbf{u}(\bar{\mathbf{w}})), \tag{4.10}$$

i.e., the continuous Jacobian matrix $A(\mathbf{u})$ expressed in terms of $\mathbf{w}$ and evaluated at the arithmetic mean $\bar{\mathbf{w}}$.

However, since

$$\frac{\bar{w}_2}{\bar{w}_1} = \frac{\sqrt{\rho_R}u_R + \sqrt{\rho_L}u_L}{\sqrt{\rho_R}+\sqrt{\rho_L}} = \tilde{u}, \tag{4.11a}$$

$$\frac{\bar{w}_3}{\bar{w}_1} = \frac{\sqrt{\rho_R}v_R + \sqrt{\rho_L}v_L}{\sqrt{\rho_R}+\sqrt{\rho_L}} = \tilde{v}, \qquad \text{and} \tag{4.11b}$$

$$\frac{\bar{w}_4}{\bar{w}_1} = \frac{\sqrt{\rho_R}H_R + \sqrt{\rho_L}H_L}{\sqrt{\rho_R}+\sqrt{\rho_L}} = \tilde{H}, \tag{4.11c}$$

say, then

$$A(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma-3)}{2}\tilde{u}^2 + \frac{(\gamma-1)}{2}\tilde{v}^2 & (3-\gamma)\tilde{u} & -(\gamma-1)\tilde{v} & (\gamma-1) \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} & 0 \\ -\tilde{u}\tilde{H} + \frac{(\gamma-1)}{2}\tilde{u}^3 + \frac{(\gamma-1)}{2}\tilde{u}\tilde{v}^2 & \tilde{H} - (\gamma-1)\tilde{u}^2 & -(\gamma-1)\tilde{u}\tilde{v} & \gamma\tilde{u} \end{pmatrix}, \tag{4.12}$$

which is precisely the Roe matrix given in [2], and clearly represents an approximation to the Jacobian (4.5). In particular, the eigenvalues of $A(\mathbf{u}_L,\mathbf{u}_R)_{\Phi_0}$ are

$$\tilde{u} \pm \tilde{a}, \tilde{u}, \tilde{u}, \tag{4.13a–d}$$

where $\tilde{u}$ is given above, and $\tilde{a}$ satisfies

$$\tilde{a}^2 = (\gamma-1)\left(\tilde{H} - \frac{1}{2}\tilde{u}^2 - \frac{1}{2}\tilde{v}^2\right), \tag{4.14}$$

as an approximation to the sound speed $a = \sqrt{\gamma p/\rho}$.

5. STEADY FLOWS

The steady equations of compressible flow are

$$\mathbf{f}_x + \mathbf{g}_y = \mathbf{0}, \tag{5.1}$$

where

$$\mathbf{f} = (\rho u, p + \rho u^2, \rho uv, u(e+p))^{\top}, \qquad \text{and} \tag{5.2}$$
$$\mathbf{g} = (\rho v, \rho vu, p + \rho v^2, v(e+p))^{\top}, \tag{5.3}$$

as before, and $(\rho, u, v, e, p) = (\rho, u, v, e, p)(x, y)$, together with

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2). \tag{5.4}$$

6. AN APPROXIMATE RIEMANN SOLVER (WEAK FORMULATION)

Steady Case

In this case, we use the formulation derived in [4] for the shallow water equations and adopt the straightforward approach, as follows. Thus, for the system (5.1), a corresponding locally linearised Riemann problem is

$$\mathbf{f}_x + M(\mathbf{f}_L, \mathbf{f}_R)_{\mathbf{\Phi}}\mathbf{f}_y = \mathbf{0}, \tag{6.1}$$

where the matrix $M(\mathbf{f}_L, \mathbf{f}_R)_{\mathbf{\Phi}}$ is a constant matrix and which is constructed in the same way that $A(\mathbf{u}_L, \mathbf{u}_R)_{\mathbf{\Phi}}$ is as outlined in Section 3. In particular, the jump condition becomes

$$\mathbf{g}(\mathbf{f}_R) - \mathbf{g}(\mathbf{f}_L) = M(\mathbf{f}_L, \mathbf{f}_R)_{\mathbf{\Phi}}(\mathbf{f}_R - \mathbf{f}_L), \tag{6.2}$$

the matrix $M(\mathbf{f}_L, \mathbf{f}_R)_{\mathbf{\Phi}}$ is an approximation to the continuous Jacobian

$$M = \frac{\partial \mathbf{g}}{\partial \mathbf{f}}, \qquad \text{such that} \tag{6.3}$$
$$M(\mathbf{f}, \mathbf{f})_{\mathbf{\Phi}} = M(\mathbf{f}), \tag{6.4}$$

and that the matrix $A_0(\mathbf{w}) = \partial \mathbf{f}_0 / \partial \mathbf{w}$ is constructed from $\mathbf{f}_0$ where $\mathbf{f}_0(\mathbf{w}) = \mathbf{f}$, as opposed to $\mathbf{u}$, where $\mathbf{w}$ is the parameter vector.

Before continuing, however, we show that the two alternative approaches contained in [4] are equivalent, and in particular, lead to a simpler derivation of the matrix

$$M_{\mathbf{\Phi}} = M(\mathbf{f}_L, \mathbf{f}_R)_{\mathbf{\Phi}}.$$

The approach outlined above which follows the procedure in [1] determines the matrix $M_{\mathbf{\Phi}} = M_{\mathbf{\Phi}_0}$ (using the canonical path), from

$$M_{\mathbf{\Phi}_0} = C_{\mathbf{\Phi}_0} B_{\mathbf{\Phi}_0}^{-1}, \qquad \text{where} \tag{6.5}$$
$$B_{\mathbf{\Phi}_0} = \int_0^1 A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds, \tag{6.6}$$
$$C_{\mathbf{\Phi}_0} = \int_0^1 M(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)))\, A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds, \tag{6.7}$$

and where

$$A_0 = \frac{\partial \mathbf{f}_0}{\partial \mathbf{w}}, \tag{6.8}$$

$$M = \frac{\partial \mathbf{g}}{\partial \mathbf{f}}, \qquad \text{with} \tag{6.9}$$

$$\mathbf{f}_0(\mathbf{w}) = \mathbf{f}(\mathbf{w}). \tag{6.10}$$

Combining (6.8) and (6.10), we have

$$A_0 = \frac{\partial \mathbf{f}}{\partial \mathbf{w}}, \tag{6.11}$$

and hence (6.11) and (6.9) imply that

$$MA_0 = \frac{\partial \mathbf{g}}{\partial \mathbf{f}}\frac{\partial \mathbf{f}}{\partial \mathbf{w}} = \frac{\partial \mathbf{g}}{\partial \mathbf{w}}. \tag{6.12}$$

Thus, the matrix in (6.5) is determined from (6.6) and (6.7) using (6.11) and (6.12), i.e.,

$$M_{\Phi_0} = \left(\int_0^1 \frac{\partial \mathbf{g}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right)\left(\int_0^1 \frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right)^{-1}. \tag{6.13}$$

On the other hand, the alternative approach is to calculate M_{Φ_0} from

$$M_{\Phi_0} = Q_{\Phi_0} P_{\Phi_0}^{-1}, \tag{6.14}$$

where the individual matrices Q_{Φ_0} and P_{Φ_0} are determined separately via the formulae

$$P_{\Phi_0} = C_P B_P^{-1}, \tag{6.15}$$

$$Q_{\Phi_0} = C_Q B_Q^{-1}. \tag{6.16}$$

The component matrices are determined using the canonical path as

$$B_P = \int_0^1 A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds, \tag{6.17}$$

$$C_P = \int_0^1 P(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)))\,A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds, \tag{6.18}$$

$$B_Q = \int_0^1 A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds, \qquad \text{and} \tag{6.19}$$

$$C_Q = \int_0^1 Q(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)))\,A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds, \tag{6.20}$$

where in this case

$$\mathbf{f}_0(\mathbf{w}) = \mathbf{u}(\mathbf{w}), \qquad \text{so that} \tag{6.21}$$

$$A_0 = \frac{\partial \mathbf{u}}{\partial \mathbf{w}}, \qquad \text{and} \tag{6.22}$$

$$P = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, \tag{6.23}$$

$$Q = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}. \tag{6.24}$$

Now, combining (6.12) with (6.13) and (6.24) gives

$$PA_0 = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{w}} = \frac{\partial \mathbf{f}}{\partial \mathbf{w}}, \qquad \text{and} \tag{6.25}$$

$$QA_0 = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{w}} = \frac{\partial \mathbf{g}}{\partial \mathbf{w}}, \tag{6.26}$$

so that (6.14)–(6.16) imply that

$$M_{\Phi_0} = Q_{\Phi_0} P_{\Phi_0}^{-1} = (C_Q B_Q^{-1})(C_P B_P^{-1})^{-1} = C_Q B_Q^{-1} B_P C_P^{-1} = C_Q C_P^{-1}, \tag{6.27}$$

since $B_P = B_Q$, and thus using (6.18) and (6.20) in (6.25) and (6.26), means that equation (6.27) becomes

$$M_{\Phi_0} = \left(\int_0^1 \frac{\partial \mathbf{g}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right)\left(\int_0^1 \frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right)^{-1}. \tag{6.28}$$

The expressions in equations (6.13) and (6.28) are identical, and thus the two approaches are identical.

Before adopting the former approach, we make the additional observation that, in view of the expression in (6.13), it is only necessary to calculate the Jacobians of $\mathbf{f}$ and $\mathbf{g}$ with respect to the parameter vector $\mathbf{w}$ in order to determine M_{Φ_0}, and this is the option that we follow in the next section. In particular, it avoids direct calculation of the Jacobian $M = \frac{\partial \mathbf{g}}{\partial \mathbf{f}}$. If the individual matrices P_{Φ_0} and Q_{Φ_0} are required, however, which is certainly true for a numerical scheme based on the system of equations (5.1) written as

$$P_{\Phi_0}\mathbf{u}_x + Q_{\Phi_0}\mathbf{u}_y = \mathbf{0}, \tag{6.29}$$

or equivalently

$$\mathbf{u}_x + \mathcal{M}_{\Phi_0}\mathbf{u}_y = \mathbf{0}, \qquad \text{where} \tag{6.30}$$

$$\mathcal{M}_{\Phi_0} = P_{\Phi_0}^{-1} Q_{\Phi_0}, \tag{6.31}$$

then (6.15)–(6.26) show that

$$\begin{aligned} P_{\Phi_0} = C_P B_P^{-1} &= \left(\int_0^1 \frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right) \\ &\quad \times \left(\int_0^1 \frac{\partial \mathbf{u}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right)^{-1}, \qquad \text{and} \end{aligned} \tag{6.32}$$

$$\begin{aligned} Q_{\Phi_0} = C_Q B_Q^{-1} &= \left(\int_0^1 \frac{\partial \mathbf{g}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right) \\ &\quad \times \left(\int_0^1 \frac{\partial \mathbf{u}}{\partial \mathbf{w}}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\,ds\right)^{-1}. \end{aligned} \tag{6.33}$$

7. APPLICATION TO STEADY, SUPERSONIC FLOWS

We now calculate the matrix M_{Φ_0} using the simplification described at the end of Section 6, as well as the individual matrices P_{Φ_0} and Q_{Φ_0}.

First, with parameter vector

$$\mathbf{w} = (w_1, w_2, w_3, w_4)^\top = (\sqrt{\rho}, \sqrt{\rho}u, \sqrt{\rho}v, \sqrt{\rho}H)^\top, \tag{7.1}$$

as in Section 4, then

$$\mathbf{u}(\mathbf{w}) = (\rho, \rho u, \rho v, e)^\top = \left(w_1^2, w_1 w_2, w_1 w_3, \frac{w_1 w_4}{\gamma} + \frac{(\gamma-1)}{2\gamma} w_2^2 + \frac{(\gamma-1)}{2\gamma} w_3^2\right)^\top, \tag{7.2}$$

$$\begin{aligned} \mathbf{f}(\mathbf{u}(\mathbf{w})) &= (\rho u, p + \rho u^2, \rho u v, u(e+p))^\top \\ &= \left(w_1 w_2, \frac{(\gamma-1)}{\gamma} w_1 w_4 + \frac{(\gamma+1)}{2\gamma} w_2^2 - \frac{(\gamma-1)}{2\gamma} w_3^2, w_2 w_3, w_2 w_4\right)^\top, \qquad \text{and} \end{aligned} \tag{7.3}$$

$$\begin{aligned} \mathbf{g}(\mathbf{u}(\mathbf{w})) &= (\rho v, \rho v u, p + \rho v^2, v(e+p))^\top \\ &= \left(w_1 w_3, w_2 w_3, \frac{(\gamma-1)}{\gamma} w_1 w_4 - \frac{(\gamma-1)}{2\gamma} w_2^2 + \frac{(\gamma+1)}{2\gamma} w_3^2, w_3 w_4\right)^\top. \end{aligned} \tag{7.4}$$

Thus,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{w}} = \begin{pmatrix} 2w_1 & 0 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ w_3 & 0 & w_1 & 0 \\ \frac{w_4}{\gamma} & \frac{(\gamma-1)w_2}{\gamma} & \frac{(\gamma-1)w_3}{\gamma} & \frac{w_1}{\gamma} \end{pmatrix}, \tag{7.5}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{w}} = \begin{pmatrix} w_2 & w_1 & 0 & 0 \\ \frac{(\gamma-1)w_4}{\gamma} & \frac{(\gamma+1)w_2}{\gamma} & -\frac{(\gamma-1)w_3}{\gamma} & \frac{(\gamma-1)}{\gamma}w_1 \\ 0 & w_3 & w_2 & 0 \\ 0 & w_4 & 0 & w_2 \end{pmatrix}, \quad \text{and} \tag{7.6}$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{w}} = \begin{pmatrix} w_3 & 0 & w_1 & 0 \\ 0 & w_3 & w_2 & 0 \\ \frac{(\gamma-1)w_4}{\gamma} & -\frac{(\gamma-1)w_2}{\gamma} & \frac{(\gamma+1)w_3}{\gamma} & \frac{(\gamma-1)w_1}{\gamma} \\ 0 & 0 & w_4 & w_3 \end{pmatrix}. \tag{7.7}$$

Now, the Jacobians (7.5)–(7.7) appear in equations (6.28), (6.32) and (6.33) as integrands, and since

$$\int_0^1 (\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds = \frac{1}{2}(\mathbf{w}_L + \mathbf{w}_R) = \bar{w}, \tag{7.8}$$

the arithmetic mean of left and right states, and all entries in these Jacobians are linear, then the approximate Jacobians are replicas of (7.5)–(7.7) but with w_i replaced by $\bar{w}_i$. The required expressions for P_{Φ_0} and Q_{Φ_0} are then

$$P_{\Phi_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma-3)\tilde{u}^2}{2} + \frac{(\gamma-1)}{2}\tilde{v}^2 & (3-\gamma)\tilde{u} & -(\gamma-1)\tilde{v} & \gamma-1 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} & 0 \\ \tilde{u}\left(\frac{(\gamma-1)}{2}\tilde{u}^2 + \frac{(\gamma-1)\tilde{v}^2}{2} - \tilde{H}\right) & \tilde{H} - (\gamma-1)\tilde{u}^2 & -(\gamma-1)\tilde{u}\tilde{v} & \gamma\tilde{u} \end{pmatrix}, \tag{7.9}$$

$$Q_{\Phi_0} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\tilde{v}\tilde{u} & \tilde{v} & \tilde{u} & 0 \\ \frac{(\gamma-1)\tilde{u}^2}{2} + \frac{(\gamma-3)}{2}\tilde{v}^2 & -(\gamma-1)\tilde{u} & (3-\gamma)\tilde{v} & \gamma-1 \\ \tilde{v}\left(\frac{(\gamma-1)\tilde{u}^2}{2} + \frac{(\gamma-1)\tilde{v}^2}{2} - \tilde{H}\right) & -(\gamma-1)\tilde{u}\tilde{v} & \tilde{H} - (\gamma-1)\tilde{v}^2 & \gamma\tilde{v} \end{pmatrix}, \tag{7.10}$$

where we denoted

$$\tilde{u} = \frac{\bar{w}_2}{\bar{w}_1} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \tag{7.11}$$

$$\tilde{v} = \frac{\bar{w}_3}{\bar{w}_1} = \frac{\sqrt{\rho_L}v_L + \sqrt{\rho_R}v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad \text{and} \tag{7.12}$$

$$\tilde{H} = \frac{\bar{w}_4}{\bar{w}_1} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}. \tag{7.13}$$

The matrices $M_{\Phi_0} = Q_{\Phi_0}P_{\Phi_0}^{-1}$ and $\mathcal{M}_{\Phi_0} = P_{\Phi_0}^{-1}Q_{\Phi_0}$ can then be determined from (7.9) and (7.10). Finally, the important quantities for a numerical scheme based on the Riemann solver of Section 6 are the average eigenvalues of M_{Φ_0} (and $\mathcal{M}_{\Phi_0}$), say $\tilde{\lambda}_i$, and these can be determined via

$$\left|Q_{\Phi_0} - \tilde{\lambda}_i P_{\Phi_0}\right| = 0, \tag{7.14}$$

to give

$$\tilde{\lambda}_i = \frac{\tilde{u}\tilde{v} \pm \tilde{a}^2\sqrt{\tilde{m}^2 - 1}}{\tilde{u}^2 - \tilde{a}^2}, \frac{\tilde{v}}{\tilde{u}}, \frac{\tilde{v}}{\tilde{u}}, \qquad i = 1,2,3,4, \tag{7.15a–d}$$

where the approximation, $\tilde{a}$, to the sound speed

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{(\gamma-1)\left(H - \frac{1}{2}(u^2+v^2)\right)} \quad \text{satisfies } \tilde{a}^2 = (\gamma-1)\left(\tilde{H} - \frac{1}{2}(\tilde{u}^2+\tilde{v}^2)\right), \tag{7.16}$$

and the approximation, $\tilde{m}$, to the Mach number $m = \frac{\sqrt{u^2+v^2}}{a}$ satisfies

$$\tilde{m}^2 = \frac{\tilde{u}^2 + \tilde{v}^2}{\tilde{a}^2}. \tag{7.17}$$

We note that the results above are equivalent to those given in [5] for the special case of an ideal gas.

8. CONCLUSIONS

We have applied the weak formulation of Roe's approximate Riemann solver to two-dimensional, compressible flows of an ideal gas in both the unsteady and steady cases, and we have shown that this leads to existing schemes. We have demonstrated the equivalence of the two alternative formulations proposed recently for the steady case, and that a simpler derivation is possible. In future work, we intend to extend these schemes to the real gas case, as well as examining the connection between the direct and weak formulations for a general system.

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